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Total Eccentricity Polynomial of Cartesian Product of Unitary Cayley Graph and Some Standard Graphs

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Abstract: In this work, the primary focus is on the investigation of the Total Eccentricity Polynomial, which is generated from the Cartesian product of Unitary Cayley graphs and important standard graphs. These graphs include path, star, wheel, friendship, full binomial tree helm, crown, $\boldsymbol{Pl_n}$, double star, and grid graphs. In addition to elucidating the development of graphs through the Cartesian product, it offers a straightforward explanation of how the polynomial is developed. In addition, it provides a detailed, step-by-step study of how eccentricity, diameter, radius, and vertex degree in each graph affect the development of the Cartesian product and the role the function plays in defining the coefficients in the Total Eccentricity Polynomial. This polynomial serves as a general tool for graphs obtained from the Cartesian product. It also paves the way for further research that uses various graph products from the literature, including other standard graphs. As a result, it presents a substantial potential for applications in related mathematical and computational domains through these innovative considerations.

Keywords: Cartesian Product; Graph Polynomials; Total Eccentricity Polynomial; Unitary Cayley Graphs; Standard Graphs; Local Structure; Product Graphs; Eccentricity Metrics.

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1. Introduction

Graph polynomials are an important feature of current graph theory because they can capture structural information in concise algebraic forms. The Total Eccentricity Polynomial has become a useful tool for discussing the eccentricity-related features of graphs and for understanding how distance-based attributes change under different graph operations. This study examines the Total Eccentricity Polynomial derived from the Cartesian product of the Unitary Cayley graph and other well-researched standard graphs. These are path graphs, star graphs, wheel graphs, friendship graphs, complete binomial trees, helm graphs, crown graphs, and the family of graphs Pl_n., double star graphs, and grid graphs. Each of these basic graphs has its own unique structural traits. When you take the Cartesian product of one of these graphs with the Unitary Cayley graph, you get new families of graphs that have a lot of different combinations of local and global attributes. By examining these combinations, one can identify overarching trends, uncover structural relationships, and formulate polynomial expressions that encapsulate the eccentricity distribution of the resultant product graphs. The Unitary Cayley graph is a well-organised object constructed on the ring of integers modulo n. The vertices are integers, and the edges connect pairs that differ by a unit modulo n. This

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structure automatically adds arithmetic features to the graph's topology, which makes it both regular and symmetrical. This kind of regularity helps determine eccentricity values, especially when the graphs are combined using the Cartesian product. The Cartesian product of two graphs, G and H, abbreviated G□H, creates a graph whose vertex set is made up of ordered pairs (g,h) with adjacency specified in a way that keeps the local structure of each graph. This action has a big effect on distance measures.

The distance between two vertices in the Cartesian product is the total of their distances in each factor graph. So, the eccentricity of a vertex in the product is the largest sum of distances to all other vertices. This depends on the eccentricities of the vertices in the individual graphs. The Total Eccentricity Polynomial is based on how diameter, radius, and vertex degree distributions affect eccentricity. This is why it is important to examine how these parameters interact in the product graph. The Total Eccentricity polynomial of a graph G, usually written as *TEP(G,x),* is the polynomial whose coefficients are the number of vertices with a certain eccentricity value. So, to find the polynomial, you need to find all the possible eccentricity values in the graph and count how many vertices have those values. For Cartesian product graphs, these eccentricity values come from combining the eccentricities of the original component graphs. This means that the polynomial is a very sensitive 3indicator of how the product's structure works. The difficulty is figuring out how the eccentricity of each vertex pair (u,v) changes when the distances of the factors are combined. The Unitary Cayley graph is a good platform for getting explicit polynomial forms when paired with standard graphs, since it often contains known eccentricity values and a predictable distance distribution. When the Unitary Cayley graph and the path graph P_n are combined, the resulting graph has a layered linear structure, with vertices connected horizontally and the path adding a vertical alignment. The eccentricity in this type of Cartesian product is based on the longest horizontal or vertical distance needed to get to the other end. The path graph has a clear set of eccentricities that increase symmetrically from the ends to the centre. These features, combined with the regularity of the Unitary Cayley graph, yield a polynomial whose coefficients reveal how the interactions between increasing and constant eccentricity patterns change.

The star graph has one central vertex that connects to all the other vertices. The Cartesian product endows the Unitary Cayley graph with a structure in which each layer has a common hub. In this case, eccentricity estimates differ because the star's dominant core vertex makes distances shorter for many vertices but much longer for leaf-to-leaf paths. The Total Eccentricity Polynomial that emerges from this shows this duality, with two main classes of eccentricity: one for vertices connected to the central star vertex and another for vertices connected to the leaves. The cycle of the wheel graph and a centre vertex make a structure that interacts with the regular arrangement of the Unitary Cayley graph in a unique way. The Cartesian product creates a graph with several cycles stacked on top of each other on arithmetic units. This makes the eccentricity more evenly spread out than it is in the star or path situations. The diameter of the wheel graph is usually modest, so the eccentricity values don't change much. This means the polynomial has fewer distinct eccentricity values, but each coefficient may have many of them. The friendship graph consists of several triangles that share a point. This creates symmetry and recurring patterns that change the distribution of eccentricity. When you take its Cartesian product with the Unitary Cayley graph, you get groups of triangular units distributed throughout the arithmetic cyclic structure. This kind of repetition allows polynomial patterns to be found by using a consistent eccentricity signature in each triangle part. The Total Eccentricity Polynomial for this combination shows repeating patterns of coefficients that come from the triangular clusters in the arithmetic grid. Complete binomial trees and helm graphs add hierarchical and ornamental structures to the mix. The recursive expansion of a complete binomial tree produces oddities that differ based on depth and branching factors.

When you combine this kind of graph with the Unitary Cayley graph, the product graph shows an increase in depth-based eccentricity together with uniform horizontal extensions. This makes a polynomial with a bigger range of eccentricity values. The helm graph, which is made by attaching pendant vertices to each vertex of a cycle, makes an oscillating eccentricity distribution that goes into the Cartesian product. This causes the eccentricity coefficients to change in size depending on the positions of the vertices relative to the cycle or pendant structure. Crown graphs and double star graphs have more uneven patterns of eccentricity. A crown graph is a bipartite graph that has densely packed oddities. It is obtained by removing perfect matching edges from a complete bipartite graph. When combined with the Unitary Cayley graph, this structure grows symmetrically, yielding polynomial coefficients that reveal both bipartition and arithmetic alignment. The double star graph has two star subgraphs connected by a central edge. The eccentricities of the graph depend on the relative sizes of the star components. The Cartesian product makes these imbalances worse, which means that the polynomial coefficients depend heavily on the diameters of the stars. This gives us many opportunities to examine how eccentricity affects asymmetry in graph products. The grid graph is a two-dimensional mesh formed as the Cartesian product of two path graphs. When you add the Unitary Cayley graph to the mix, you get a multidimensional grid that sits on top of the arithmetic structure. The eccentricity values in these kinds of graphs are often spread over a wide range because distances can be very large in both the horizontal and planar directions.

In this situation, the Total Eccentricity Polynomial is among the most complicated of the families analysed, as it has several distinct eccentricity values and large changes in the coefficients. The polynomial shows how eccentricity can grow across multiple dimensions and how combining a structured arithmetic graph with a geometric grid strengthens distance-based traits. The main contribution of the work across these typical graph families is to show that the Total Eccentricity Polynomial behaves predictably when the features of the individual graphs are well established. The study connects the distribution of eccentricity to structural characteristics like diameter, radius, and vertex degree [6]; [7]. This makes it possible to find closed-form polynomial formulas for Cartesian product graphs that can be used in other situations. This not only improves our theoretical understanding of invariants based on eccentricity, but also provides useful methods for analysing massive composite graphs. This work has consequences that go beyond just maths. Eccentricity metrics are essential in network analysis, optimisation, communication design, and data structure assessment. Understanding how eccentricity behaves when graph products are used can help you build complex networks that retain desirable traits such as strength, low communication delay, hierarchical layering, and even distribution. The Total Eccentricity Polynomial functions as an algebraic descriptor that can facilitate additional enquiries in algorithmic graph theory, spectral graph analysis, chemical graph theory, and computational modelling. By providing clear methods for finding this polynomial for a large number of Cartesian product graphs, the work enables investigation of other graph products, such as tensor, strong, or lexicographic products, as well as other types of standard graphs.

2. Methodology

The total eccentricity polynomial is a graph invariant that captures the distribution of vertex eccentricities. Each vertex contributes a monomial whose exponent equals its eccentricity, and summing these monomials over all vertices produces a polynomial encoding the eccentricity profile of the graph [3]. For a graph G with vertex set $\mathrm{V(G)}$ and eccentricity $\mathrm{e(u)}$ of a vertex u, the total eccentricity polynomial is defined as:

$$\mathrm{TEP(G)} = \sum_{u \in V(G)} x^{e(u)}.$$

This representation allows the coefficient of x^k to be interpreted as the number of vertices having eccentricity k, which is useful in distance-based analysis of graphs in chemical and algebraic graph theory. Evaluating the derivative of $TEP(G)$ at $x = 1$ yields the total eccentricity of the graph, i.e., the sum of eccentricities of all vertices. These ideas and variations of eccentricity-based polynomials are discussed in works such as Ashrafi et al. [1] and Ghorbani et al. [2], as well as related papers on eccentricity-based indices. Cayley graphs, constructed from group elements and generating sets, exhibit vertex transitivity and high symmetry, making them fundamental in combinatorics, group theory, and computer science applications such as interconnection networks and parallel architectures [10]. Their universality allows modelling of scalable distributed systems, data centres, and multi-agent coordination structures [4]. Unitary Cayley graphs form a specialised subclass of the ring. $\mathbb{Z}_n$, where edges connect integers modulo ndiffering by units (elements coprime to n). First defined by Dejter and Giudici, these graphs possess rich algebraic properties, including Hamiltonicity under certain conditions and applications in domination theory. The Cartesian product of graphs combines graphs while preserving key properties: the product of Cayley graphs remains a Cayley graph, with vertices as pairs and edges respecting individual adjacencies. This paper examines the *TEP* of Cartesian products of unitary Cayley graphs and standard graphs such as paths and grids, yielding explicit formulas that reveal how algebraic symmetry interacts with linear or radial structures [5].

3. Preliminaries

Definition 3.1: Given a finite group G and a symmetric connector set $\mathrm{S \subseteq G - e}$, the Cayley graph, denoted $\mathrm{Cay(G, S)}$, is the graph with $\mathrm{V = G}$ and $\mathrm{E = \{(x, y) \in V \times V : x^{-1}y \in S}$ (i.e, $\mathrm{y^{-1} = xs}$ for some $\mathrm{s \in S}$.) } [6]-[7].

Definition 3.2: The total eccentricity polynomial of a graph G is defined as $\mathrm{TEP(G)} = \sum_{u \in V(G)} x^{e(u)}$, where, e(u)=eccentricity of u [10].

Definition 3.3: The Unitary Cayley graph X_n has a vertex set $V(X_n) = \{0,1,2, \dots, n-1\}$ and edge set $E(X_n) = \{(a, b) \mid \gcd(a, b) = u_n\}$ where $u_n = \{0,1\}$ = units of the ring [8]; [9].

Figure 1 illustrates the first nine unitary Cayley graphs:

Figure 1: The first nine unitary Cayley graphs

Definition 3.4: The cartesian product of two graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and where (x, y) and (x', y') are adjacent if and only if $x = x'$ and y is adjacent to y' in H, or $y = y'$ and x is adjacent to x' in G [11]. The degree and eccentricity of a vertex (x, y) of the Cartesian product graph is, $\deg(x, y) = \deg(x) + \deg(y)$, and $e(x, y) = e(x) + e(y)$ [12].

4. Main Results

Theorem 4.1: For the Cartesian product of the Unitary Cayley graph X_n and path graph P_m, the Total Eccentricity polynomial (TEP) is:

$$TEP(X_n \square P_m) = \begin{cases} 2n\sum_{i=1}^{\frac{m}{2}} x^{d_u+m-i} & \text{if m is even} \\ 2n\sum_{i=1}^{\left\lfloor\frac{m}{2}\right\rfloor-1} x^{d_u+m-i} + nx^{d_u+\frac{m-1}{2}} & \text{if m is odd} \end{cases}$$

Where d_u =diameter of the Unitary Cayley graph.

Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3, \dots, n\}$, $V(P_n) = \{v_j, \text{ for } j = 1,2,3, \dots, m\}$ and $V(X_n \square P_m) = \{u_i v_j |\ i = 1,2,3, \dots, n; j = 1,2,3, \dots, m\}$ be the vertex set of the Unitary Cayley graph $X_{n,}$ the path graph $P_{m,}$ and the Cartesian product of the Unitary Cayley graph X_n and path graph P_m, $(X_n \square P_m)$ respectively [11]. Eccentricity of each vertex in the unitary Cayley graph X_n is, $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n:

$e(v_j) = \{m - j \text{ for } j = l, m - (l - 1)$, where $1 \le l \le \left\lceil\frac{m}{2}\right\rceil$
Since, $e(u_i v_j) = e(u_i) + e(v_j)$, $e(u_i v_j) = d_u + m - j$ for $j = l, m - (l - 1)$, where $1 \le l \le \left\lceil\frac{m}{2}\right\rceil$.
By the definition of TEP:

$$TEP(X_n \square P_m) = \sum_{u_i v_j \in X_n \square P_m} x^{e(u_i v_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_i v_1)} + x^{e(u_i v_2)} + \dots + x^{e(u_i v_m)}\right]$$

$$= \sum_{u_i \in V(Xn)} \left[x^{d_u+ m-1} + x^{d_u+ m-2} + x^{d_u+ m-3} + \dots + x^{d_u+ m-3} + x^{d_u+ m-2} + x^{d_u+ m-1}\right]$$

If m is even:

$$= 2n\left[x^{d_u+ m-1} + x^{d_u+ m-2} + x^{d_u+ m-3} + \dots + x^{d_u+ \frac{m}{2}}\right]$$

$$TEP(X_n \square P_m) = 2n\sum_{i=2}^{\frac{m}{2}} x^{d_u+ m-i}$$

If m is odd:

$$= n\left[2\left[x^{d_u+ m-1} + x^{d_u+ m-2} + x^{d_u+ m-3} + \dots + x^{d_u+ \frac{m-3}{2}}\right] + x^{d_u+ \frac{m-1}{2}}\right]$$

$$TEP(X_n \square P_m) = 2n\sum_{i=2}^{\left\lfloor\frac{m}{2}\right\rfloor-1} x^{d_u+ m-i} + nx^{d_u+ \frac{m-1}{2}}$$

Theorem 4.2: For the Cartesian product of the Unitary Cayley graph X_n and star graph $K_{1,m}$, the Total Eccentricity polynomial (TEP) is, $TEP(X_n \square K_{1,m}) = n\left[x^{d_u+1} + mx^{d_u+2}\right]$, where d_u =diameter of the Unitary Cayley graph.

Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3, \dots, n\}$, $V(P_n) = \{v_j, \text{ for } j = 0,1,2,3, \dots, m\}$ and $V(X_n \square K_{1,m}) = \{u_i v_j |\ i = 1,2,3, \dots, n; j = 0,1,2,3, \dots, m\}$ be the vertex set of the Unitary Cayley graph $X_{n,}$ the star graph $K_{1,m}$, shown below, and the

Cartesian product of the Unitary Cayley graph X_n and star graph $K_{1,m}$, $(X_n \square K_{1,m})$ respectively (Figure 2). Eccentricity of each vertex in the unitary Cayley graph X_n is, $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n:

$$e(v_j) = \begin{cases} 1 & \text{if } j = 0 \\ 2 & \text{if } j = 1,2,3,\dots,m \end{cases}$$

Since, $e(u_i v_j) = e(u_i) + e(v_j)$, $e(u_i v_j) = \begin{cases} d_u + 1 & \text{if } j = 0 \\ d_u + 2 & \text{if } j = 1,2,3,\dots,m \end{cases}$

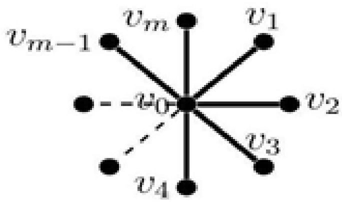

Figure 2: Star graph $K_{1,m}$ with central vertex v_0 and peripheral vertices $\{v_1, v_2, \dots, v_m\}$

By the definition of TEP:

$$TEP(X_n \square K_{1,m}) = \sum_{u_i v_j \in X_n \square K_{1,m}} x^{e(u_i v_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_i v_0)} + x^{e(u_i v_1)} + \dots + x^{e(u_i v_m)}\right]$$

$$= \sum_{u_i \in X_n} \left[x^{d_u+1} + \underbrace{x^{d_u+2} + \dots + x^{d_u+2}}_{\text{m times}}\right]$$

$$TEP(X_n \square K_{1,m}) = n\left[x^{d_u+1} + mx^{d_u+1}\right]$$

Remark: It is observed that, $TEP(X_n \square K_{1,m}) = n\left[x^{d_u+1} + mx^{d_u+1}\right] = TEP(X_n \square W_m) = TEP(X_n \square F_m)$, Where, W_m is the wheel graph with vertex set, $V(W_m) = \{v_j, \text{ for } j = 0,1,2,3,\dots,m\}$ and F_m is the Friendship graph F_m (or Dutch windmill graph or m-fan), $V(F_m) = \{v_j, \text{ for } j = 0,1,2,3,\dots,m\}$.

Theorem 4.3: For the Cartesian product of the Unitary Cayley graph X_n and a complete binomial tree, B_m , the Total Eccentricity polynomial (TEP) is, $TEP(X_n \square B_m) = n\left[x^{d_u+1} + 2x^{d_u+m} + +2^2x^{d_u+m+1} + \dots + +2^{m-2}x^{d_u+m}\right]$, where, d_u = diameter of the Unitary Cayley graph.

Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3,\dots,n\}$, $V(B_m) = \{v_j, \text{ for } j = 2,3,\dots,2^m\}$ and $V(X_n \square B_m) = \{u_i v_j | \ i = 1,2,3,\dots,n; j = 2,3,\dots,2^m\}$ be the vertex set of the Unitary Cayley graph X_n, Complete binomial tree B_m and the Cartesian product of the Unitary Cayley graph X_n and Complete binomial tree B_m, shown below for m = 4, $(X_n \square B_m)$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n (Figure 3).

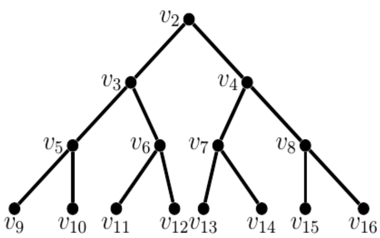

Figure 3: Binary tree with labelled vertices $\{v_2, \dots, v_{16}\}$

$$e(v_j) = \begin{cases} m-1 & \text{if } j = 2 \\ m+k & \text{if } 2^{k+1}+1 \le j \le 2^k \end{cases}$$

Since, $e(u_iv_j) = e(u_i) + e(v_j)$, $e(u_iv_j) = \begin{cases} d_u + m - 1 & \text{if } j = 2 \\ d_u + m + k & \text{if } 2^{k+1}+1 \le j \le 2^k \end{cases}$, where $k = m - 2$

By the definition of TEP:

$$TEP(X_n \,\Box\, B_m) = \sum_{u_iv_j \in X_n \Box B_m} x^{e(u_iv_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_iv_1)} + x^{e(u_iv_2)} + \cdots + x^{e(u_iv_{2^k})}\right]$$

$$= \sum_{u_i \in X_n} \left[x^{d_u+m-1} + x^{d_u+m} + x^{d_u+m} + \underbrace{x^{d_u+m+1} + \cdots + x^{d_u+m+1}}_{2^2 \text{ times}} + \cdots + \underbrace{x^{d_u+2m-2} + \cdots + x^{d_u+2m-2}}_{2^{m-2} \text{ times}}\right]$$

$$TEP(X_n \,\Box\, B_m) = n\left[x^{d_u+1} + 2x^{d_u+m} + +2^2 x^{d_u+m+1} + \cdots + 2^{m-2} x^{d_u+m}\right]$$

Theorem 4.4: For the Cartesian product of the Unitary Cayley graph X_n and the Helm graph H_m, the Total Eccentricity polynomial (TEP) is, $TEP(X_n \,\Box\, H_m) = n[x^{d_u+2} + m[x^{d_u+3} + x^{d_u+4}]]$, where, d_u = diameter of the Unitary Cayley graph.
Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3,\ldots,n\}$, $V(H_m) = \{v_j, \text{ for } j = 0,1,2,3,\ldots,2m\}$ and $V(X_n \,\Box\, H_m) = \{u_iv_j| \ i = 1,2,3,\ldots,n; j = 0,1,2,3,\ldots,2m\}$ be the vertex set of Unitary Cayley graph X_n, the Helm graph H_m, shown below, and the Cartesian product of the Unitary Cayley graph X_n and the Helm graph H_m, $(X_n \,\Box\, H_m)$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is, $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n (Figure 4):

$$e(v_j) = \begin{cases} 2, & \text{if } j = 0, \\ 3, & \text{if } 1 \le j \le m, \\ 4, & \text{if } m+1 \le j \le 2m \end{cases}$$

Since $e(u_iv_j) = e(u_i) + e(v_j)$, $e(u_iv_j) = \begin{cases} d_u + 2, & \text{if } j = 0, \\ d_u + 3, & \text{if } 1 \le j \le m, \\ d_u + 4, & \text{if } m+1 \le j \le 2m \end{cases}$

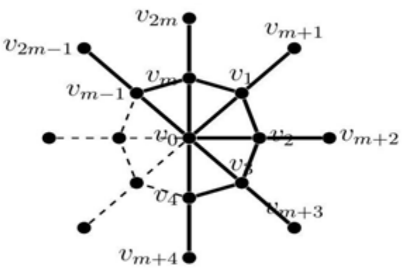

Figure 4: Helm graph Hm

By the definition of TEP:

$$TEP(X_n \,\Box\, H_m) = \sum_{u_iv_j \in X_n \Box H_m} x^{e(u_iv_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_iv_0)} + x^{e(u_iv_1)} + \cdots + x^{e(u_iv_{2m})}\right]$$

$$= \sum_{u_i \in X_n} \left[x^{d_u+2} + \underbrace{x^{d_u+3} + \cdots + x^{d_u+3}}_{m \text{ times}} + \underbrace{x^{d_u+4} + \cdots + x^{d_u+4}}_{m \text{ times}}\right]$$

$TEP(X_n \square H_m) = n[\, x^{d_u+2} + m[x^{d_u+3} + x^{d_u+3}]]$

Theorem 4.5: For the Cartesian product of the Unitary Cayley graph X_n and the Crown graph CW_m, the Total Eccentricity polynomial (TEP) is:

$$TEP(X_n \square CW_m) = \begin{cases} nm\left[x^{d_u+\frac{n}{2}+1} + x^{d_u+\frac{n}{2}+2}\right] \text{ when m is even,} \\ nm\left[x^{d_u+\frac{n+1}{2}} + x^{d_u+\frac{n+3}{2}}\right] \text{ when m is odd,} \end{cases}$$

Where, d_u = diameter of the Unitary Cayley graph.

Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3,\dots,n\}$, $V(CW_m) = \{v_j, \text{ for } j = 1,2,3,\dots,2m\}$ and $V(X_n \square CW_m) = \{u_iv_j | \ i = 1,2,3,\dots,n; j = 1,2,3,\dots,2m\}$ be the vertex set of the Unitary Cayley graph X_n, the Crown graph CW_m, shown below, and the Cartesian product of the Unitary Cayley graph X_n and the Crown graph CW_m, $(X_n \square CW_m)$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n (Figure 5).

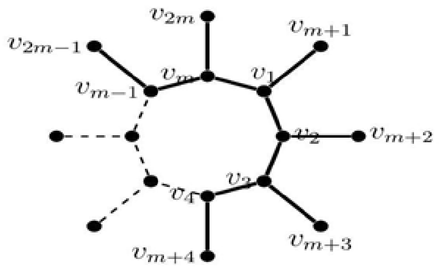

Figure 5: Crown graph CWm

Case 1: When m is even:

$$e(v_j) = \begin{cases} \frac{n}{2} + 1, & \text{if } 1 \le j \le m, \\ \frac{n}{2} + 2, & \text{if } m+1 \le j \le 2m \end{cases}, \quad e(u_iv_j) = \begin{cases} d_u + \frac{n}{2} + 1, & \text{if } 1 \le j \le m, \\ d_u + \frac{n}{2} + 2, & \text{if } m+1 \le j \le 2m \end{cases}$$

By the definition of TEP:

$$TEP(X_n \square CW_m) = \sum_{u_iv_j \in X_n \square CW_m} x^{e(u_iv_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_iv_1)} + x^{e(u_iv_2)} + \cdots + x^{e(u_iv_{2m})}\right]$$

$$= \sum_{u_i \in X_n} \left[\underbrace{x^{d_u+\frac{n}{2}+1} + \cdots + x^{d_u+\frac{n}{2}+1}}_{\text{m times}} + \underbrace{x^{d_u+\frac{n}{2}+2} + \cdots + x^{d_u+\frac{n}{2}+2}}_{\text{m times}}\right]$$

$$TEP(X_n \square CW_m) = nm\left[x^{d_u+\frac{n}{2}+1} + x^{d_u+\frac{n}{2}+2}\right]$$

Case 2: When m is odd:

$$e(v_j) = \begin{cases} \frac{n+1}{2}, & \text{if } 1 \le j \le m, \\ \frac{n+3}{2}, & \text{if } m+1 \le j \le 2m \end{cases}, \quad e(u_iv_j) = \begin{cases} d_u + \frac{n+1}{2}, & \text{if } 1 \le j \le m, \\ d_u + \frac{n+3}{2}, & \text{if } m+1 \le j \le 2m \end{cases}$$

By the definition of TEP:

$$\mathrm{TEP}(X_n \,\square\, CW_m) = \sum_{u_i v_j \in X_n \square CW_m} x^{e(u_i v_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_i v_1)} + x^{e(u_i v_2)} + \cdots + x^{e(u_i v_{2m})}\right]$$

$$= \sum_{u_i \in X_n} \left[\underbrace{x^{d_u+\frac{n+1}{2}} + \cdots + x^{d_u+\frac{n+1}{2}}}_{\text{m times}} + \underbrace{x^{d_u+\frac{n+3}{2}} + \cdots + x^{d_u+\frac{n+3}{2}}}_{\text{m times}}\right]$$

$$TEP(X_n \,\square\, CW_m) = nm\left[x^{d_u+\frac{n+1}{2}} + x^{d_u+\frac{n+3}{2}}\right]$$

Theorem 4.6: For the Cartesian product of the Unitary Cayley graph X_n and the Pl_n graph $Pl_m, m \geq 6$, the Total Eccentricity polynomial (TEP) is, $\mathrm{TEP}(X_n \,\square\, Pl_m) = n[2x^{d_u+1} + (m-2)x^{d_u+2}]$, where, d_u = diameter of the Unitary Cayley graph (Figure 6).

Proof: Let $V(X_n) = \{u_i, \text{for } i = 1,2,3,\dots,n\}$, $V(Pl_m) = \{v_j, \text{for } j = 1,2,3,\dots,m\}$ and $V(X_n \,\square\, Pl_m) = \{u_i v_j | \, i = 1,2,3,\dots,n; j = 1,2,3,\dots,m\}$ be the vertex set of Unitary Cayley graph X_n, the Pl_n graph $Pl_m, m \geq 6$, shown below, and the Cartesian product of the Unitary Cayley graph X_n and the Pl_n graph Pl_m, $(X_n \,\square\, Pl_m)$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n:

$$e(v_j) = \begin{cases} 1, & \text{if } j = 1,2 \\ 2, & \text{otherwise.} \end{cases}, \quad e(u_i v_j) = \begin{cases} d_u + 1, & \text{if } j = 1,2 \\ d_u + 2, & \text{otherwise.} \end{cases}$$

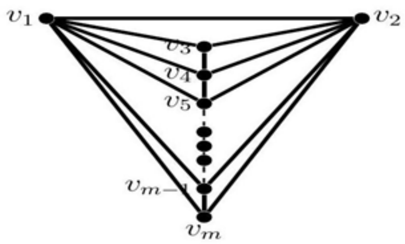

Figure 6: Pln graph

By the definition of TEP:

$$\mathrm{TEP}(X_n \,\square\, Pl_m) = \sum_{u_i v_j \in X_n \square Pl_m} x^{e(u_i v_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_i v_0)} + x^{e(u_i v_1)} + \cdots + x^{e(u_i v_m)}\right]$$

$$= \sum_{u_i \in X_n} \left[x^{d_u+1} + x^{d_u+1} + \underbrace{x^{d_u+2} + \cdots + x^{d_u+2}}_{\text{m-2times}}\right]$$

$$\mathrm{TEP}(X_n \,\square\, Pl_m) = n[2x^{d_u+1} + (m-2)x^{d_u+2}]$$

Theorem 4.7: For the Cartesian product of the Unitary Cayley graph X_n and the Double Star graph $S_{p,q}$, the Total Eccentricity polynomial (TEP) is, $\mathrm{TEP}(X_n \,\square\, S_{p,q}) = n[2x^{d_u+3} + (p+q-2)x^{d_u+3}]$, where, d_u = diameter of the Unitary Cayley graph (Figure 7).

Proof: Let $V(X_n) = \{u_i, \text{for } i = 1,2,3,\dots,n\}$, $V(S_{p,q}) = \{v_j, \text{for } j = 1,2,3,\dots,m\}$ and $V(X_n \,\square\, S_{p,q}) = \{u_i v_j | \, i = 1,2,3,\dots,n; j = 1,2,3,\dots,m\}$ be the vertex set of Unitary Cayley graph X_n, the Double Star graph $S_{p,q}$ and the Cartesian product of the Unitary

Cayley graph X_n and the Double Star graph $S_{p,q}$ shown below, $(X_n \square S_{p,q})$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n:

$$e(v_j) = \begin{cases} 2, & \text{if } j = 1, p+1 \\ 3, & \text{otherwise.} \end{cases}, e(u_i v_j) = \begin{cases} d_u + 2, & \text{if } j = 1, p+1 \\ d_u + 3, & \text{otherwise.} \end{cases}$$

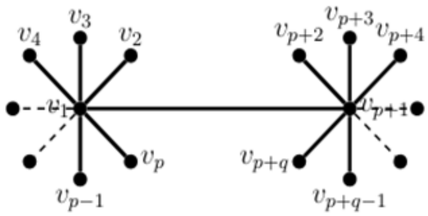

Figure 7: Double star graph Sp, q

By the definition of TEP:

$$TEP(X_n \square S_{p,q}) = \sum_{u_i v_j \in X_n \square S_{p,q}} x^{e(u_i v_j)}$$

$$= \sum_{u_i \in V(Xn)} \left[x^{e(u_i v_1)} + x^{e(u_i v_2)} + \cdots + x^{e(u_i v_{p+q})}\right]$$

$$= \sum_{u_i \in X_n} \left[x^{d_u+2} + \underbrace{x^{d_u+3} + \cdots + x^{d_u+3}}_{p-1\text{times}} + x^{d_u+2} + \underbrace{x^{d_u+3} + \cdots + x^{d_u+3}}_{q-1\text{times}}\right]$$

$$TEP(X_n \square S_{p,q}) = n\left[2x^{d_u+2} + (p+q-2)x^{d_u+3}\right]$$

Theorem 4.8: For the Cartesian product of the Unitary Cayley graph X_n and the Grid graph $P_p \square P_q$, the Total Eccentricity polynomial (TEP) is:

$$TEP(X_n \square (P_p \square P_q)) = \begin{cases} 4x^{d_u+p+q-2} + \sum_{r=2}^{\frac{p}{2}} 4x^{d_u+p+q-r-1} + \sum_{s=2}^{\frac{q}{2}} 4x^{d_u+p+q-s-1} + \sum_{r=2}^{\frac{p}{2}} \sum_{s=2}^{\frac{p}{2}} x^{d_u+p+q-r-s} \\ \qquad \text{if } p \text{ and } q \text{ are even} \\ 4x^{d_u+p+q-2} + \sum_{r=2}^{\frac{p}{2}} 4x^{d_u+p+q-r-1} + \sum_{s=2}^{\lfloor \frac{q}{2} \rfloor -1} 4x^{d_u+p+q-s-1} + 2x^{d_u+p+\frac{q-1}{2}} \\ + \sum_{r=2}^{\lfloor \frac{p}{2} \rfloor} \sum_{s=2}^{\lfloor \frac{q}{2} \rfloor} x^{d_u+p+q-r-s} + 2\sum_{s=2}^{\lfloor \frac{q}{2} \rfloor} (d_n+4) x^{d_u+q+\lfloor \frac{p}{2} \rfloor - s} + x^{d_u+\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor} \\ \qquad \text{if } p \text{ is even and } q \text{ are odd} \\ 4x^{d_u+p+q-2} + \sum_{r=2}^{\lfloor \frac{p}{2} \rfloor -1} 4x^{d_u+p+q-r-1} + 2x^{d_u+q+\frac{p-1}{2}} + \sum_{s=2}^{\lfloor \frac{q}{2} \rfloor -1} 4x^{d_u+p+q-s-1} + 2x^{d_u+p+\frac{q-1}{2}} \\ + \sum_{r=2}^{\lfloor \frac{p}{2} \rfloor} \sum_{s=2}^{\lfloor \frac{q}{2} \rfloor} x^{d_u+p+q-r-s} + 2\sum_{s=2}^{\lfloor \frac{q}{2} \rfloor} x^{d_u+q+\lfloor \frac{p}{2} \rfloor - s} + 2\sum_{r=2}^{\lfloor \frac{p}{2} \rfloor} x^{d_u+p+\lfloor \frac{q}{2} \rfloor - r} + x^{d_u+\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor} \\ \qquad \text{if } p \text{ and } q \text{ are odd} \end{cases}$$

Where, d_u = diameter of the Unitary Cayley graph.

Proof: Let $V(X_n) = \{u_i, \text{ for } i = 1,2,3,\ldots,n\}$, $V(P_p \square P_q) = \{v_j w_k, \text{ for } j = 1,2,3,\ldots,p;\ k = 1,2,3,\ldots,q\}$ and $V(X_n \square S_{p,q}) = \{u_i v_j w_k | \ i = 1,2,3,\ldots,n; j = 1,2,3,\ldots,p; k = 1,2,3,\ldots,q\}$ be the vertex set of the Unitary Cayley graph X_n, the grid graph $P_p \square P_q$ and the Cartesian product of the Unitary Cayley graph X_n and the grid graph $P_p \square P_q$ shown below, $(X_n \square (P_p \square P_q))$ respectively. Eccentricity of each vertex in the unitary Cayley graph X_n is $e(u_i) = d_u$, where, d_u= diameter of Unitary Cayley graph X_n (Figure 8).

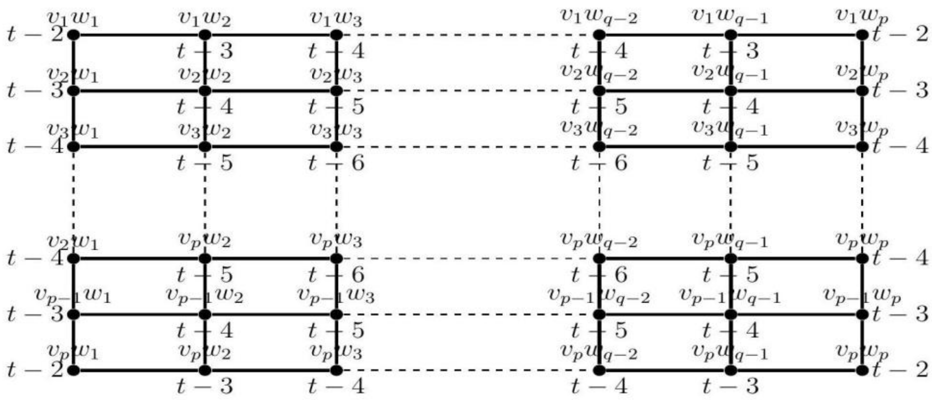

Figure 8: Grid graph

Where $t = p + q$ in the above grid graph:

$e(w_k) = \{q - s, \text{ if } k = s, q - (s-1)$

Where, $1 \le s \le \left\lceil \frac{q}{2} \right\rceil$

$$e(v_j w_k) = \begin{cases} p - r + q - s, & \text{for } j = r, p-(r-1) \text{ where, } 1 \le r \le \left\lceil \frac{p}{2} \right\rceil, \text{ and} \\ & k = s, q-(s-1) \text{ where, } 1 \le s \le \left\lceil \frac{q}{2} \right\rceil \end{cases}$$

$$e(u_i v_j w_k) = \begin{cases} d_u + p - r + q - s, & \text{for } j = r, p-(r-1) \text{ where, } 1 \le r \le \left\lceil \frac{p}{2} \right\rceil, \text{ and} \\ & k = s, q-(s-1) \text{ where, } 1 \le s \le \left\lceil \frac{q}{2} \right\rceil \end{cases}$$

$$TEP(X_n \,\square\, (P_p \,\square\, P_q)) = \sum_{u_i v_{j \in X_n \square P_p \square P_q}} x^{e(u_i v_j w_k)}$$

$$= \sum_{u_i \in V(Xn)} \begin{bmatrix} x^{e(u_i v_1 w_1)} + x^{e(u_i v_1 w_2)} + \cdots + x^{e(u_i v_1 w_q)} + \\ x^{e(u_i v_2 w_1)} + x^{e(u_i v_2 w_2)} + \cdots + x^{e(u_i v_2 w_q)} + \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ x^{e(u_i v_p w_1)} + x^{e(u_i v_p w_2)} + \cdots + x^{e(u_i v_p w_q)} \end{bmatrix}$$

$$= \sum_{u_i = \in V(Xn)} \begin{bmatrix} x^{d_u+p+q-2} + x^{d_u+p+q-3} + x^{d_u+p+q-4} + \cdots + x^{d_u+p+q-4} + x^{d_u+p+q-3} + x^{d_u+p+q-2} + \\ x^{d_u+p+q-3} + x^{d_u+p+q-4} + x^{d_u+p+q-5} + \cdots + x^{d_u+p+q-5} + x^{d_u+p+q-4} + x^{d_u+p+q-3} + \\ x^{d_u+p+q-4} + x^{d_u+p+q-5} + x^{d_u+p+q-6} + \cdots + x^{d_u+p+q-6} + x^{d_u+p+q-5} + x^{d_u+p+q-4} + \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \\ x^{d_u+p+q-4} + x^{d_u+p+q-5} + x^{d_u+p+q-6} + \cdots + x^{d_u+p+q-6} + x^{d_u+p+q-5} + x^{d_u+p+q-4} + \\ x^{d_u+p+q-3} + x^{d_u+p+q-4} + x^{d_u+p+q-5} + \cdots + x^{d_u+p+q-5} + x^{d_u+p+q-4} + x^{d_u+p+q-3} + \\ x^{d_u+p+q-2} + x^{d_u+p+q-3} + x^{d_u+p+q-4} + \cdots + x^{d_u+p+q-4} + x^{d_u+p+q-3} + x^{d_u+p+q-2} \end{bmatrix}$$

When p and q are even:

$$TEP\left(X_n \,\square\, (P_p \,\square\, P_q)\right) = 4x^{d_u+p+q-2} + 4\sum_{r=2}^{\frac{p}{2}} x^{d_u+p+q-r-1} + 4\sum_{s=2}^{\frac{q}{2}} x^{d_u+p+q-s-1} + \sum_{r=2}^{\frac{p}{2}} \sum_{s=2}^{\frac{q}{2}} x^{d_u+p+q-s-r}$$

When p is even, and q is odd:

$$\mathrm{TEP}\left(X_n \,\square\, (P_p \,\square\, P_q)\right) = 4x^{d_u+p+q-2} + 4\sum_{r=2}^{\frac{p}{2}} x^{d_u+p+q-r-1} + 4\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor-1} x^{d_u+p+q-s-1} + 2x^{d_u+p+\frac{q-1}{2}} + \sum_{r=2}^{\lfloor\frac{p}{2}\rfloor}\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor} x^{d_u+p+q-s-r} + 4\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor} x^{d_u+\lfloor\frac{p}{2}\rfloor+q-s} + x^{d_u+\lfloor\frac{p}{2}\rfloor+\lfloor\frac{q}{2}\rfloor}$$

When p and q are odd:

$$\mathrm{TEP}\left(X_n \,\square\, (P_p \,\square\, P_q)\right) = 4x^{d_u+p+q-2} + 4\sum_{r=2}^{\lfloor\frac{p}{2}\rfloor-1} x^{d_u+p+q-r-1} + 2x^{d_u+q+\frac{p-1}{2}} + 4\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor-1} x^{d_u+p+q-s-1} + 2x^{d_u+p+\frac{q-1}{2}} + \sum_{r=2}^{\lfloor\frac{p}{2}\rfloor}\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor} x^{d_u+p+q-s-r} + 4\sum_{s=2}^{\lfloor\frac{q}{2}\rfloor} x^{d_u+\lfloor\frac{p}{2}\rfloor+q-s} + 4\sum_{r=2}^{\lfloor\frac{p}{2}\rfloor} x^{d_u+\lfloor\frac{q}{2}\rfloor+p-r} + x^{d_u+\lfloor\frac{p}{2}\rfloor+\lfloor\frac{q}{2}\rfloor}$$

5. Conclusion

This paper develops a clear and systematic method for analysing the Total Eccentricity Polynomial of Cartesian products involving Unitary Cayley graphs and various standard graphs. It thoroughly examines how fundamental graph properties such as eccentricity, diameter, radius, and vertex degree influence the construction of Cartesian product graphs and determine the coefficients that appear in their associated polynomials. By providing a unified framework that applies broadly across different graph products, this work not only deepens the theoretical understanding of these mathematical objects but also establishes practical guidelines for their computation. The insights gained herein open new directions for future research, encouraging the exploration of other graph products and their polynomial invariants within the mathematical literature. Moreover, the approach demonstrated offers strong potential for applications in network analysis, chemistry, and computer science, where graph structures play a critical role. Ultimately, this study contributes both theoretical advancements and useful computational tools, enhancing the capability to analyse and interpret complex graph structures via polynomial measures, thereby enriching the broader field of graph theory and its interdisciplinary connections.

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